Math 250 2.3 Strategies for Evaluating Limits Analytically

Strategy #0: Know when to evaluate limits by direct substitution and when this strategy fails. [See page 2.]

Strategy #1: If necessary, build an intuitive understanding of the function by using numerical and graphical methods to decide if the limit might exist and its approximate value. (Caution: this is not an analytical or exact method, just a way to estimate.)

Strategy #2: Break the limit of the sum, difference, product, quotient, and/or root of functions into the sum, difference, product, quotient, and/or root of limits of those functions. Use properties of limits in Theorems. [See page 2.]

CAUTION:

$\frac{0}{0}$ is called an indeterminate form, because it does not determine anything except that the strategy failed!

$\frac{0}{0}$ can resolve to be a number, zero, or $\pm\infty$. $\frac{0}{0}$ means we are not finished finding the limit.

Do NOT EVER write $\frac{0}{0}$ as a final answer, because it will earn NO credit.

Cancel a common factor appears in the next four strategies. If you want the limit as $x \to a$, then the common factor you want to cancel will be $(x-a)$, because $\frac{(x-a)}{(x-a)}$ is causing the $\frac{0}{0}$.

Strategy #3: Factor a rational expression to **cancel a common factor** $(x-a)$, and then substitute.

Strategy #4: Rationalize either the numerator or denominator of an expression containing radicals; **cancel a common factor** $(x-a)$, and then substitute.

Strategy #5: Simplify any complex fractions (fractions within fractions) by multiplying by the LCD top and bottom (or by dividing), factor completely, **cancel a common factor** $(x-a)$, and then substitute.

Strategy #6: Multiply the numerator and denominator by a "conjugate"-like expression to get a difference of squares, and then use a trig identity or factor and **cancel a common factor** $(x-a)$, and then substitute.

Strategy #7: Use the Squeeze Theorem: If

- $f(x) \le g(x) \le h(x)$ for all x in an open interval containing c, (except possibly c itself), AND
- $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$,

then $\lim_{x\to a} g(x) = L$ also.

Step 1: Find two functions, $f(x)$ and $h(x)$, so that $f(x) \le g(x) \le h(x)$ for values of x near a.

Step 2: Show that the two functions have the same limit as x approaches a: $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$

Step 3: Conclude that $\lim_{x\to a} g(x) = L$

Math 250 2.3 Evaluating Limits Analytically Using Substitution

> If the behavior of the function near $x = a$ is the same as the function value at $f(a)$, then the limits and the function value are the same $\left(\lim_{x\to a} f(x) = f(a)\right)$ and we can evaluate the limit by using direct substitution.

Theorem 2.2 Limits of Linear Functions: If a, *b* and *m* are real numbers and $f(x) = mx + b$, then:
$\lim_{x\to a} f(x) = f(a) = ma + b$

If $m = 0$, then $\lim_{x\to a} b = b$ (constant function)

If $m = 1$ and $b = 0$, then $\lim_{x\to a} x = a$

Theorem 2.3 Limit Laws: If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, *c* is a real number, and $m > 0$, $n > 0$ are integers such that n/m is in lowest terms, then:

a) $\lim_{x\to a}[b \cdot f(x)] = b \cdot \lim_{x\to a} f(x)$ (scalar multiple)

b) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ (sum or difference)

c) $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ (product)

d) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (quotient)

e) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ (power)

f) $\lim_{x\to a}[f(x)]^{n/m} = \left(\lim_{x\to a} f(x)\right)^{n/m}$ (fractional power)

Fractional powers are tricky. If *m* is odd, then the statement is true.
But if *m* is even, e.g. $(f(x))^{3/4} = \sqrt[4]{f(x)}$, this radical is not defined unless $f(x) \geq 0$ for *x* values near *a*.

Similarly, $\lim_{x\to a^+}[f(x)]^{n/m} = \left(\lim_{x\to a^+} f(x)\right)^{n/m}$ is true if *m* is odd, or if *m* is even and $f(x) \geq 0$ for $x > a$.

$\lim_{x\to a^-}[f(x)]^{n/m} = \left(\lim_{x\to a^-} f(x)\right)^{n/m}$ is true if *m* is odd, or if *m* is even and $f(x) \geq 0$ for $x < a$.

Theorem 2.4 Polynomial and Rational Functions If *p* and *q* are polynomials and *a* is a real constant, then

a) $\lim_{x\to a} p(x) = p(a)$

b) $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, so long as $q(a) \neq 0$.

Theorem (not in Briggs) Composition of two functions
If f and g are functions with $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then $\lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right] = f(L)$

Theorem (not in Briggs) Trigonometric functions
If *a* is a real number in the domain of the trig function, then:

a) $\lim_{x\to a} \sin x = \sin a$

b) $\lim_{x\to a} \cos x = \cos a$

c) $\lim_{x\to a} \tan x = \tan a$

d) $\lim_{x\to a} \cot x = \cot a$

e) $\lim_{x\to a} \sec x = \sec a$

f) $\lim_{x\to a} \csc x = \csc a$

Math 250 Evaluating Limits Analytically

CAUTION: Write the limit notation in each step of your work until you have taken the limit!

Practice:

Evaluate the limit exactly, using analytical methods.

1) $\lim_{x\to 6}\left(6x^5-\frac{4}{3}x^2+x-\pi\right)$

2) $\lim_{x\to -2}\frac{x^3+8}{x+2}$

3) $\lim_{x\to 5}\frac{x-5}{x^2-25}$

4) $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

5) $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

6) $\lim_{x\to \frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}$

Given that $\lim_{x\to a} f(x)=2$ and $\lim_{x\to a} g(x)=3$, find the following limits.

7) $\lim_{x\to a} 2f(x)\cdot g(x)$

8) $\lim_{x\to a}\left[\frac{4f(x)+(g(x))^{5/6}}{(g(x))^7}-\frac{f(x)\cdot g(x)}{8}\right]$

9) Evaluate the limit using the Squeeze Theorem.

$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

a. Show that $-|x|\le x\sin\left(\frac{1}{x}\right)\le |x|$ for x near 0.

b. Take the limit of this compound inequality.

10) Find constants b and c in the polynomial $p(x)=x^2+bx+c$ such that $\lim_{x\to 2}\frac{p(x)}{x-2}=6$. Are the constants unique? ("Unique" means "one of a kind"; if the answer is unique, there's only one possible solution.)

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① $\lim_{x \to 6} \left(6x^5 - \frac{4}{3}x^2 + x - \pi\right)$

$= 6 \cdot \left[\lim_{x \to 6} x\right]^5 - \frac{4}{3}\left[\lim_{x \to 6} x\right]^2 + \left[\lim_{x \to 6} x\right] - \pi$ using Limit Laws

$= 6 \cdot (6)^5 - \frac{4}{3}(6)^2 + 6 - \pi$

$= 46656 - 48 + 6 - \pi$

$= \boxed{46614 - \pi}$

The symbol π is exact because its meaning includes all of its ∞-many decimal places.

② $\lim_{x \to -2} \frac{x^3 + 8}{x + 2} = \frac{(-2)^3 + 8}{(-2) + 2} = \frac{0}{0}$ indeterminate ☹

factor $x^3 + 8 = (x+2)(x^2 - 2x + 4)$

$= \lim_{x \to -2} \frac{\cancel{(x+2)}(x^2 - 2x + 4)}{\cancel{(x+2)}}$

$= \lim_{x \to -2} (x^2 - 2x + 4)$

$= (-2)^2 - 2(-2) + 4$

$= 4 + 4 + 4$

$= \boxed{12}$

③ $\lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \frac{5 - 5}{5^2 - 25} = \frac{0}{0}$ indeterminate ☹

factor $x^2 - 25 = (x+5)(x-5)$

$= \lim_{x \to 5} \frac{\cancel{x-5}}{(x+5)\cancel{(x-5)}}$

$= \lim_{x \to 5} \frac{1}{x+5}$

$= \frac{1}{5+5} = \boxed{\frac{1}{10}}$

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④ $$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \frac{\sqrt{3+1}-2}{3-3} = \frac{0}{0}$$ indeterminate ☹

Rationalize the numerator

$$= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)}{x-3} \underbrace{\frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}}_{\text{This fraction} = 1}$$

conjugate has same terms but opposite sign between

Foil only the numerator → goal is to cancel a common factor $(x-3)$, so leave denominator factored.

$$= \lim_{x\to 3} \frac{x+1+2\sqrt{x+1}-2\sqrt{x+1}-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{(\sqrt{x+1}+2)}$$

$$= \frac{1}{\sqrt{3+1}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{2+2}$$

$$= \boxed{\frac{1}{4}}$$

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(5) $$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \frac{\frac{1}{2+0} - \frac{1}{2}}{0} = \frac{0}{0}$$ indeterminate ☹

Method 1: mult numerator & denominator by LCD $2(2+x)$.

$$= \lim_{x\to 0} \frac{\frac{1}{\cancel{2+x}} \cdot 2\cancel{(2+x)} - \frac{1}{\cancel{2}} \cdot \cancel{2}(2+x)}{\frac{x}{1} \cdot 2(2+x)}$$

$$= \lim_{x\to 0} \frac{2-(2+x)}{2x(2+x)}$$

$$= \lim_{x\to 0} \frac{2-2-x}{2x(2+x)}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}}{2\cancel{x}(2+x)}$$

$$= \lim_{x\to 0} \frac{-1}{2(2+x)}$$

$$= \frac{-1}{2(2+0)}$$

$$= \frac{-1}{2 \cdot 2}$$

$$= \boxed{\frac{-1}{4}}$$

Method 2: Add fractions in numerator using LCD, then divide using multiply by reciprocal.

$$= \lim_{x\to 0} \frac{\frac{1}{2+x} \cdot \frac{2}{2} - \frac{1}{2}\frac{(2+x)}{(2+x)}}{x}$$

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$$= \lim_{x\to 0} \frac{\dfrac{2-(2+x)}{2(2+x)}}{\dfrac{x}{1}}$$

$$= \lim_{x\to 0} \frac{2-(2)-x}{2(2+x)} \cdot \frac{1}{x}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}}{2\cancel{x}(2+x)}$$

$$= \lim_{x\to 0} \frac{-1}{2(2+x)}$$

$$= \boxed{-\frac{1}{4}}$$ as before

(6) $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x}$

$\tan\frac{\pi}{4}=1$
$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
gives $\frac{0}{0}$ ☹
keep working

$= \displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\left(1-\frac{\sin x}{\cos x}\right)}{(\sin x - \cos x)}$

rewrite $\tan x = \frac{\sin x}{\cos x}$

$= \displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\left(1-\frac{\sin x}{\cos x}\right)\cdot \cos x}{(\sin x - \cos x)\cdot \cos x}$

clear complex fractions

$= \displaystyle\lim_{x\to\frac{\pi}{4}} \frac{(\cos x - \sin x)}{(\sin x - \cos x)\cdot \cos x}$

$= \displaystyle\lim_{x\to\frac{\pi}{4}} \frac{-\cancel{(\sin x - \cos x)}}{\cancel{(\sin x - \cos x)}\cdot \cos x}$

factor out -1
cancel common factor

$= \displaystyle\lim_{x\to\frac{\pi}{4}} \frac{-1}{\cos x}$

$= \dfrac{-1}{\cos\frac{\pi}{4}}$

$= \dfrac{-1}{(\sqrt{2}/2)}$

$= \dfrac{-2}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}$

$= \boxed{-\sqrt{2}}$

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$\lim_{x \to a} f(x) = 2$ and $\lim_{x \to a} g(x) = 3$ are given.

7) $\lim_{x \to a} 2 \cdot f(x) \cdot g(x)$

$= 2 \cdot \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right]$

$= 2 \cdot 2 \cdot 3$

$= \boxed{12}$

8) $\lim_{x \to a} \left[\dfrac{4f(x) + g(x)^{5/6}}{(g(x))^7} - \dfrac{f(x) \cdot g(x)}{8} \right]$

$= \dfrac{4 \cdot \left[\lim_{x \to a} f(x)\right] + \left[\lim_{x \to a} g(x)\right]^{5/6}}{\left[\lim_{x \to a} g(x)\right]^7} - \dfrac{\left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right]}{8}$

$= \dfrac{4 \cdot 2 + (3)^{5/6}}{3^7} - \dfrac{2 \cdot 3}{8}$

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$$= \frac{8 + 3^{5/6}}{2187} - \frac{6}{8}$$

$$= \frac{8 + \sqrt[6]{3^5}}{2187} - \frac{3}{4}$$

$$= \frac{8 + \sqrt[6]{243}}{287} - \frac{3}{4}$$

find a common denominator

$$= \frac{4\left(3 + \sqrt[6]{243}\right) - 3(287)}{287 \cdot 4}$$

$$= \frac{12 + \sqrt[6]{243} - 861}{1148}$$

$$= \boxed{\frac{\sqrt[6]{243} - 849}{1148}}$$

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9 Evaluate $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ using the Squeeze Theorem

a) If $f(x) \le g(x) \le h(x)$ becomes $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

(The question gives us $f(x) = -|x|$

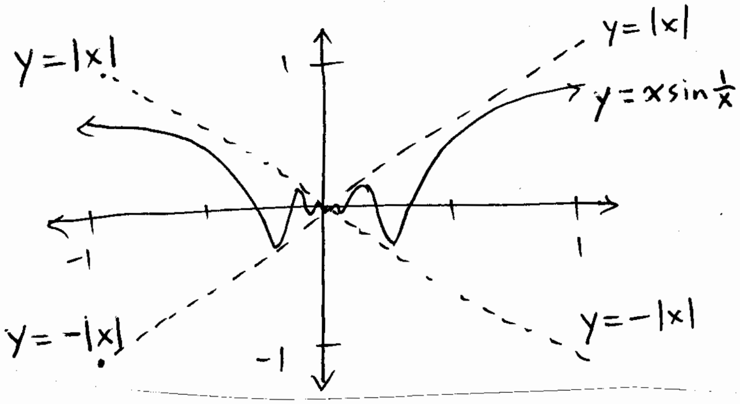
and $g(x) = |x|$

BUT... asks us to confirm the inequality!

If we consider the graphs: GC $y_1 = abs(x)$ ← - - - $\boxed{MATH}$ ▷ $\boxed{NUM}$ 1 abs(

$y_2 = -abs(x)$

$y_3 = x\sin(1/x)$



It does appear that $|x|$ is always above $x\sin\left(\frac{1}{x}\right)$ and $-|x|$ is always below $x\sin\left(\frac{1}{x}\right)$ so that

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

looks like it's probably true.

But $x\sin\left(\frac{1}{x}\right)$ is very strange (oscillation) as $x\to 0$

So we need to do this work analytically to be certain.

Recall that $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$

if $x = 5$ then $|5| = 5 = x$

if $x = -5$ then $|-5| = -(-5) = 5 = -x$

We will use this piecewise definition of absolute value AND we will split our work into 2 cases.

case #1: $x \ge 0$

case #2: $x < 0$

case #1 $x \geq 0$ means $|x| = x$

so our desired outcome $-|x| \leq x \sin\frac{1}{x} \leq |x|$

becomes $-x \leq x \sin\frac{1}{x} \leq x$

case #2 $x < 0$ means $|x| = -x$

so our desired outcome $-|x| \leq x \sin\frac{1}{x} \leq |x|$

becomes $x \leq x \sin\frac{1}{x} \leq -x$

Recall

- range of $\sin\theta$ is $[-1,1]$

  so $-1 \leq \sin\theta \leq 1$ for any θ

  so $-1 \leq \sin\frac{1}{x} \leq 1$ for $\theta = \frac{1}{x}$

- if we multiply an inequality by a negative #, we reverse the inequality

  ex. $-2x < 4$

  $$\left(-\frac{1}{2}\right) \cdot (-2x) > (4) \cdot \left(-\frac{1}{2}\right)$$

  $$x > -2$$

  ex. $-1 < -2x < 4$

  $\frac{1}{2} > x > -2$ which means

  $-2 < x < \frac{1}{2}$

case #1 $x \geq 0$ $\quad -1 \leq \sin\frac{1}{x} \leq 1$

mult by x, which is positive $\Rightarrow$ do not change the direction of the inequality

$$-x \leq x \sin\frac{1}{x} \leq x$$

which is $-|x| \leq x \sin\frac{1}{x} \leq |x|$

our desired outcome for case #1.

case #2 $x < 0$ $\quad -1 \leq \sin\frac{1}{x} \leq 1$

mult by x, which is negative $\Rightarrow$ change the direction of the inequality!

$$-x \ge x\sin\frac{1}{x} \ge x$$

$$x \le x\sin\frac{1}{x} \le -x$$

which means $-|x| \le x\sin\frac{1}{x} \le |x|$ when $x<0$!

This is our desired outcome for case #2. ☺

To summarize:

If $f(x) = -|x|$ and $h(x) = |x|$

and our desired function $g(x) = x\sin\left(\frac{1}{x}\right)$

in case #1 $x \ge 0$ $\quad -|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

means $\quad f(x) \le g(x) \le h(x)$

in case #2 $x \le 0$ $\quad -|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

means $\quad f(x) \le g(x) \le h(x)$

so the first if-statement (hypothesis) is confirmed. ☺

b) Take the limit of this compound inequality means "show that the second if-statement (hypothesis) of the Squeeze Theorem is also true.

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

$$\underbrace{\lim_{x\to 0} -|x|}_{\text{direct substitution!}} \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} |x|$$

$$-|0| \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le |0|$$

$$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$$

The Squeeze Theorem works <u>ONLY</u> if these two limits are the same number. We have squeezed our desired limit between two limits which are the same.

So... $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \boxed{0}$

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(10) Find b and c so that $\lim_{x\to 2} \frac{x^2+bx+c}{x-2} = 6$.

Are these constants unique?

Strategy: Try substitution:

$$\lim_{x\to 2} \frac{2^2+b(2)+c}{2-2} = \frac{4+2b+c}{0}$$ but it's supposed to be 6.

What if it's not $\frac{\#}{0}$ but $\frac{0}{0}$?

Then we seek a common factor which is 0 as $x\to 2$.
That factor would be $(x-2)$.

So we want $\lim_{x\to 2} \frac{\cancel{(x-2)}(\text{other factor})}{\cancel{(x-2)}} = 6$

so the other factor....
must be $(1x-\#)$ because its x^2, not ax^2.
so let's call that # .... d?

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x-d)}{\cancel{(x-2)}} = 6$$

$$2-d = 6$$

$$2-6 = d$$

$$-4 = d$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}} = 6$$

$2+4 = 6$ yes!

so $(x-2)(x+4) = x^2+bx+c$

$x^2+4x-2x-8 = x^2+bx+c$

$x^2+2x-8 = x^2+bx+c$

so $\boxed{b=2 \quad c=-8}$

There are other factors one could pair with $(x-2)$ to cancel a common factor, but only one that makes the limit 6.
$\boxed{\text{These constants are unique.}}$ (the only answer)